QUALITATIVE PHASE-PLANE ANALYSIS OF DYNAMICS OF ELECTRO-THERMAL PROCESSES IN THERMISTOR

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Analytical relations are examined which can be useful for study, by the phase-plane method, of nonlinear self-excited oscillations in an electric circuit containing a thermistor and a linear reactive element.

The electrothermal processes occurring in a thermistor after it has been connected into an electric circuit are well known to owe their peculiarities, above all, to the nonlinearity of its current—voltage characteristic with a range of negative differential resistance. Several studies have dealt with the so-called relay effect which occurs upon such a switching. In [1], specifically, was noted the possibility of a self-ocillatory process taking place in a circuit which contains a thermistor and a capacitor for the first time. The essence of this phenomenon has been discussed most extensively in [2], on the basis of thorough delving into the nature of these oscillations and with a mathematical apparatus for analysis of their stability. The linear model of a thermistor was used there for analysis of the oscillations. Such an approach is quite effective in the solution of problems pertaining to buildup and decay of oscillations nearly sinusoidal in form.

The linear theory does not reveal, however, whether nonsinusoidal oscillations can occur in such a circuit and remain stable. Nevertheless, the very existence of nonharmonic oscillations has been established experimentally [2] and the general trend of their dependence on the circuit parameters experimentally studied there. An analytical description of nonlinear oscillations is still sought.

In this report will be presented results of a qualitative study of the dynamics of electrothermal processes in an electric circuit containing a thermistor, a study made by methods of the theory of oscillations and the theory of bifurcation [3, 4], a study supplementing [2] and revealing some peculiarities of these processes.

The qualitative analysis will begin with consideration of the fundamental equation of a thermistor

$$C_V \frac{d\Delta T}{dt} = P_T - H\Delta T. \quad (1)$$

It has already been demonstrated [5] that an accurate enough approximate solution to this equation can be obtained, after the hypothesis of a "hot" gas of charge carriers in the thermistor

$$\Delta T = \Delta T_e + \Delta T_p + \Delta T_0 \quad (2)$$

has been introduced. On the basis of this condition, Eq. (1) can be written as

$$C_V \frac{d\Delta T_e}{dt} + C_V \frac{d\Delta T_p}{dt} = P_T - H(\Delta T_e + \Delta T_p + \Delta T_0). \quad (3)$$

According to [5], the temperature of the crystal lattice T_L is related to the thermistor current through the equality

$$\Delta T_L = T_L - T_L^0 = \frac{I_T - I_0}{N_L}. \quad (4)$$

In order to find an expression relating the temperature of the gas of charge carriers to the electrical parameters of the thermistor, we use the results of [6]. According to the concepts about the kinetics of nonequilibrium processes occurring in semiconductors with an S-

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form current–voltage characteristic presented in that study, the temperature T_e of the electron (charge carrier) gas is related to the electric field intensity ε in the semiconductor through the proportion

$$T_e \sim \mathrm{E}^{\frac{2}{n-m}}. \quad (5)$$

In order for the current–voltage characteristic to be an S-curve, it is necessary that n be smaller than m. This condition will be fulfilled the cause of momentum relaxation in the semiconductor is scattering by impurities (m = 3/2) and the cause of energy relaxation in the semiconductor is piezoelectric scattering by acoustic phonons (n = 1/2).

We cannot assert that such a pattern of scattering prevails in a thermistor, but assuming that it does will lead to the relation

$$T_e \sim \mathrm{E}_T^{-2}, \quad (6)$$

implying that the electron temperature increases as the electric field intensity decreases. The magnitude of ΔT_e can be found by expanding the left-hand side of the expression

$$\frac{1}{T_p^0 + \Delta T_e} \sim \mathrm{E}_T^2 \quad (7)$$

into a Taylor series in ΔT_e in the vicinity of the lattice temperature T_L° and retaining the linear approximation

$$T_p - \Delta T_e \sim \mathrm{E}_T^2. \quad (8)$$

Upon introduction of the proportionality factor N_e, one can write

$$\Delta T_e \cong T_p^0 - N_e U_T^2. \quad (9)$$

We will now consider an electric circuit consisting of a linear active four-pole network with a thermistor connected at the input and a linear reactive element connected at the output. Let the latter be a capacitance C_C.

According to fundamental principles of the theory of linear four-pole networks, the relation between input and output parameters of such a network are

$$U_T = AU_c + BI_c + L, \quad (10)$$

$$I_T = CU_c + DI_c + M. \quad (11)$$

The power generated in the thermistor is obviously

$$P_T = I_T U_T = (AU_c + BI_c + L)(CU_c + DI_c + M). \quad (12)$$

Inserting expressions (4), (9) and (12) into Eq. (3), we obtain, after appropriate transformations,

$$(\gamma_2 U_c + \lambda_2 I_c + \beta_2)\frac{dI_c}{dt} = \alpha_1 U_c + \beta_1 I_c + \gamma_1 U_c I_c + \lambda_1 I_c^2 + \delta_1 U_c^2 + \eta_1, \quad (13)$$

where

$$\alpha_1 = CL + AM - \frac{H}{N_p} C - 2HN_e AL;$$

$$\beta_1 = 2\frac{N_e C_V}{C_C} LA - \frac{C_V C}{N_p C_C} + BM + DL - \frac{H}{N_p} D + 2BLN_e H;$$

$$\gamma_1 = AD + BC + 2N_e HAB + 2N_e \frac{C_V}{C_C} A^2;$$

$$\lambda_1 = +2N_e \frac{C_V}{C_C} AB + BD + N_e HB^2; \quad \delta_1 = AC + N_e HA^2;$$

$$\eta_1 = ML - \frac{H}{N_p} M + N_e HL^2 + H\frac{I_c}{N_p} - HT_p^0 - H\Delta T_0;$$

$$\beta_2 = \frac{C_V}{N_p} D - 2N_e LB; \quad \lambda_2 = -2N_e C_V B^2; \quad \gamma_2 = -2N_e C_V AB.$$

We have taken into consideration here that

$$C_e \frac{dU_c}{dt} = I_c; \quad C_C \frac{d^2U_c}{dt^2} = \frac{dI_c}{dt}.$$

Letting arbitrarily

$$U_c = x, \quad C_C \frac{dU_c}{dt} = y,$$

we transform Eq. (13) to the system of equations

$$\frac{dy}{dt} = \frac{\alpha_1 x + \beta_1 y + \gamma_1 xy + \lambda_1 y^2 + \delta_1 x^2 + \eta_1}{\beta_2 + \gamma_2 x + \lambda_2 y}; \quad C_C \frac{dx}{dt} = y. \tag{14}$$

The system of Eqs. (14) is an autonomous dynamic one of the second order with an analytic right-hand side. It can be examined by methods and procedures of qualitative analysis in the phase plane. For convenience, we will reduce system (14) to a simpler form by eliminating the free term η_1. For this we determine the roots of the function

$$f(x) = \delta_1 x^2 + \alpha_1 x + \eta_1 \equiv \delta (x - K_1)(x - K_2).$$

Denoting these roots as K_1 and K_2, we have

$$K_{1,2} = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\delta_1\eta_1}}{2\delta_1}. \tag{15}$$

Introducing now the new variable

$$z = x - K_1,$$

we rewrite system (14) as

$$y' = \frac{-\delta_1 (K_2 - K_1) z + (\beta_1 + \gamma_1 K_1) y + \gamma_1 zy + \lambda_1 y^2 + \delta_1 z^2}{\frac{1}{C_C}[(\beta_2 + \gamma_2 K_1) y + \gamma_2 zy + \lambda_2 y^2]} = \frac{Q(z, y)}{P(z, y)}, \tag{16}$$

where $y' = dy/dz$.

We note that the physical system under consideration here (four-pole network with capacitor and thermistor) is a priori known to be a nonconservative one, as a consequence of which its singular points (equilibrium states) are gross ones.

According to the theory of nonlinear oscillations [3], gross equilibrium states corresponding to points in the phase plane where simultaneously $P(z, y) = 0$ and $Q(z, y) = 0$ are characterized by the signs of Δ and σ at these points.

Limiting our concern in this case to the character of only two singular points (out of four possible ones in system (14)) for which $y = 0$ or $I_C = 0$, which is equivalent, we have

at point K(0, 0)

$$\Delta_K = \begin{vmatrix} P_z'(0, 0) & P_y'(0, 0) \\ Q_z'(0, 0) & Q_y'(0, 0) \end{vmatrix} = \delta_1 (K_2 - K_1)(\beta_2 + \gamma_2 K_1) \frac{1}{C_C}, \quad \sigma_K = P_z'(0, 0) + Q_y'(0, 0) = \beta_1 + \gamma_1 K_1, \tag{17}$$

and

at point $S(K_2 - K_1, 0)$

$$\Delta_S = \begin{vmatrix} P_z'(K_2 - K_1, 0) & P_y'(K_2 - K_1, 0) \\ Q_z'(K_2 - K_1, 0) & Q_y'(K_2 - K_1, 0) \end{vmatrix} = -\delta_1 (K_2 - K_1)[\gamma_2 (K_2 - K_1) + (\beta_2 + \gamma_2 K_1)] \frac{1}{C_C}, \quad \sigma_S = P_z'(K_2 - K_1, 0) + Q_y'(K_2 - K_1, 0) = \beta_1 + \gamma_1 K_2. \tag{18}$$

Upon inserting specific values of the electric circuit parameters into these expressions, one can determine whether a given point is a node, a focus, or a saddle.

TABLE 1. Bifurcation of Dynamic System Due to Change in Stability of Complex Focus

Signs of L_1, σ'	$\sigma < 0$	$\sigma = 0$	$\sigma > 0$
$L_1<0$, $\sigma'>0$	Focus stable, no cycle	Focus stable, no cycle	Focus unstable, cycle stable
$L_1<0$, $\sigma'<0$	Focus unstable, cycle stable	Same	Focus stable, no cycle
$L_1>0$, $\sigma'>0$	Focus stable, cycle unstable	Focus unstable, no cycle	Focus unstable, no cycle
$L_1>0$, $\sigma'<0$	Focus unstable, no cycle	Same	Focus stable, cycle unstable

In order to determine the possibility of a limit cycle existing in the phase plane of Eq. (16), or, in other words, the possibility of self-excited oscillations occurring in the circuit with a thermistor and a linear capacitance, it is necessary to determine for this equation its so-called first Lyapunov parameter L_1 [4]

$$L_1 = -\frac{\pi}{4(\beta_2+\gamma_2K_1)}\frac{\delta_1}{[\delta_1(K_2-K_1)(\beta_2+\gamma_2K_1)]^{3/2}}\{\delta_1(K_2-K_1)^2\lambda_2(\gamma_2+\lambda_1)-\gamma_1(\beta_2+\gamma_2K_1)[\lambda_1(K_2-K_1)+\beta_2+\gamma_2K_1]\}. \tag{19}$$

One must also know, for this purpose, the sign of the quantity

$$\sigma_K = \beta_1 + \gamma_1 K_1$$

and that of its first derivative with respect to any of the circuit parameters (e.g., C_C).

After the signs of L_1, σk, and $(\sigma k)'_{C_C}$ have been determined, the question as to whether a limit cycle exists can be answered with the aid of a table.

In order to demonstrate that in a circuit containing a thermistor and a linear capacitance with an active linear four-pole network there can occur self-excited oscillations, we consider the simplest electric circuit consisting of a thermistor connected in parallel across a linear resistance and a capacitance in series. Such a circuit has already been studied experimentally [2] in detail.

Starting with the obvious relations

$$U_T = U_c, \quad I_T = -\frac{U_c}{R} - I_c + \frac{E}{R}, \tag{20}$$

we determine the coefficients of the four-pole network

$$A = 1, \quad C = -\frac{1}{R}, \quad D = -1; \quad M = \frac{E}{R}, \quad B = L = 0.$$

The dynamic processes in this circuit are described by the differential equation

$$y' = \frac{\alpha_{11}x + \beta_{11}y + \gamma_{11}xy + \delta_{11}x^2 + \eta_{11}}{\beta_{21}y}, \tag{21}$$

where

$$\alpha_{11} = -\frac{1}{R}\left(E + \frac{H}{N_p}\right); \quad \beta_{11} = -\frac{H}{N_p} - \frac{C_V}{N_pC_CR};$$

$$\gamma_{11} = -2N_e\frac{C_V}{C_C} + 1; \quad \delta_{11} = \frac{1}{R} - N_eH;$$

$$\eta_{11} = \frac{H}{N_p}\left(I_0 - \frac{E}{R}\right) - HT_p^0 - H\Delta T_0; \quad \beta_{21} = \frac{C_V}{N_pC_C}.$$

The roots K_1 and K_2 are evaluated according to expression (15), viz.

$$K_{1,2} = \left\{\frac{1}{R}\left(E + \frac{H}{N_p}\right) \pm \sqrt{\frac{1}{R^2}\left(E+\frac{H}{N_p}\right)^2 - 4\left(\frac{1}{R} - N_eH\right)\left[\frac{H}{N_p}\left(I_0 - \frac{E}{R}\right) - HT_p^0 - H\Delta T_0\right]}\right\} \times \tag{22}$$

$$\times \left\{2\left(\frac{1}{R} - N_eH\right)\right\}^{-1}.$$

By matching the parameters of the thermistor and of the other circuit elements, also the conditions of heat transfer, one can make $\delta_{11} < 0$, $\alpha_{11} < 0$, $\eta_{11} > 0$, and $\gamma_{11} > 0$, if the roots K_1 and K_2 are real. In that case $K_1 \neq K_2$, because system (21) is nonconservative (with the roots equal, the system would be conservative). We will assume, for specificity, that $K_1 > 0 > K_2$. The sign of the determinant Δ are then $\Delta_K > 0$ at point K and $\Delta_S < 0$ at point S.

Therefore, point S is always a saddle and the character of point K is determined by the sign of the discriminant $\sigma_K^2 - 4\Delta_K$. Point K is a focus when $\sigma_K^2 - 4\Delta_K < 0$ and a node in the opposite case.

Let us select the circuit parameter C_C so that at point K will be simultaneously satisfied by the conditions

$$\sigma_K = \beta_{11} + \gamma_{11}K_1 > 0 \text{ and } \sigma_K^2 - 4\Delta_K < 0.$$

Then the first Lyapunov parameter at point K will be

$$L_1 = \frac{\pi}{4}\,\frac{\delta_{11}\gamma_{11}\beta_{21}}{[\delta_{11}(K_2 - K_1)\beta_{21}]^{3/2}} < 0,$$

and, since $(\sigma_k)'_{C_C} > 0$, the table indicates the existence of a stable limit cycle in system (21). This means that, with the parameters of the given electric circuit appropriately matched, there will occur stable nonlinear self-excited oscillations.

We note that the stability at point K can be changed by a decrease of parameter C_C, which will make the limit cycle vanish and the focus become stable. Decreasing C_C further will eventually convert the stable focus to a stable node.

This peculiarity of the oscillatory process, well reflected in the proposed mathematical model, corresponds closely to the manner in which the transient process depends on the magnitude of the capacitance according to practical observations [2].

The second peculiarity of the transient process, which can also be observed in practice, is that the conditions for occurrence of self-excited oscillations depend strongly on the conditions of heat transfer from the thermistor to the ambient medium. This dependence is also evident in the proposed mathematical model, where H appears as a component of terms in the expressions for Δ and σ.

Furthermore, there exists an analogy between the phase portrait of an actual dynamic system "capacitance—linear resistance—thermistor" and the phase portrait of the dynamic system described by Eq. (21). One can ascertain this by comparing the experimental curves of study [2] with the results of analysis of a mathematical model of the (21) kind in study [7].

On the basis of the preceding comments, one can assume that the results of this study will be of definite interest not only as a supplement to the overall study of oscillatory processes in circuits with thermistors [2] but also as a useful tool for solving various practical problems concerning the application of thermistors in automatic control systems, in measuring systems, and in several other kinds of systems.

NOTATION

T, total temperature of a thermistor, °K; T_L, temperature of the thermistor lattice, °K; T_e, temperature of the hot gas of charge carriers, °K; T_L°, initial temperature of a thermistor, °K; T_0, ambient temperature, °K; $\Delta T_e = T_e - T_L$; $\Delta T_L = T_L - T_L^\circ$; $\Delta T_0 = T_L^\circ - T_0$; H, dissipation coefficient for a thermistor, W/°C; C_V, specific heat of a thermistor, W·sec/°C; N_L (A/°C), N_e (°C/B²), proportionality factors; ε, electric field intensity in a thermistor, V/m; P_T, power generated in a thermistor by passing current, W; I_T, thermistor current, A; I_0, thermistor current at t = +0, A; U_T, thermistor voltage, V; U_C, voltage across the capacitance, V; C_C, capacitance, F; E, circuit supply voltage, V; A, B, C, D, L, M, coefficients of a linear active four-pole network; R, linear resistance Ω; and t, time, sec.

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METHOD OF REDUCTION TO THE ORDINARY DIFFERENTIAL EQUATIONS OF L. V. KANTOROVICH AND A GENERAL METHOD FOR THE SOLUTION OF MULTIDIMENSIONAL HEAT-TRANSFER EQUATIONS

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A method is proposed for the solution of multidimensional heat-transfer problems, representing a further elaboration and generalization of projection methods.

The mathematical investigation of heat- and mass-transfer processes in various kinds of heat-exchange equipment is known to require the solution of complex multidimensional problems. The advent of the Ritz and Bubnov–Galerkin methods for the solution of problems in the variational and differential formulations, respectively, set the stage for the development of a powerful trend in applied mathematics, viz., projection methods [1, 2], and afforded the conceptual possibility of solving a broad category of multidimensional problems. However, even in cases where the theory guarantees convergence of the indicated methods, a sufficiently accurate solution is obtainable, as a rule, for a large number n of pertinent parameters. This fact, in turn, means the application of computing hardware. Familiar difficulties may also be encountered in connection with the onset of instability and, accordingly, a loss of accuracy of the solution with increasing value of n (contrary to theory), up to the point of complete divergence of the process [1]. Coping with these difficulties by refinement of the coordinate functions through their orthogonalization, compliance with boundary conditions, etc., is not too effective in the general case, because all of these procedures are implemented on a distinctly a priori choice of basis. A cardinal solution of the general problem can be achieved by seeking to obtain reliable (not a priori!) functional information with essential reliance on the original mathematical statement of the problem. A first step in this direction is offered by the method of reduction to Kantorovich–Vlasov ordinary differential equations [3, 4]. In this method the constant coefficients involved in the Ritz (Galerkin) procedure are superseded by functional coefficients depending on one of the arguments of the problem, i.e., the required solution $u(\bar{x})$ of an N-dimensional problem is represented in the form

$$u(\bar{x}) = F(\bar{K}(x_k),\ \bar{\varphi}(\bar{x})). \tag{1}$$

Here $\bar{\varphi}(\bar{x}) = \{\varphi_m(\bar{x})\}_{m=1}^{l}$ is a vector function of a vector argument $\bar{x} = (x_1, x_2, \ldots, x_N)$, the components of which are basic functions selected a priori; $\bar{K}(x_k) = \{K_j(x_k)\}_{j=1}^{n}$, vector function of a variable x_k, the components of which are evaluated deterministically from the one-dimensional problem; and F, function characterizing the form of representation of the solution, i.e., its structure. It is customarily assumed in projection methods that

$$u(\bar{x}) = \sum_{m=1}^{n} K_m(x_k)\, \varphi_m(\bar{x}). \tag{1a}$$

This approach improved the convergence of the solution in comparsion with the Ritz and Bubnov–Galerkin procedures. However, because of the intuitive choice of functional informa-

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